

Decay of Fourier Coefficients

Note Title

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Suppose $f \in C^1$. Then $\widehat{f}(n) = i\widehat{f}(n)$.

Hence $\widehat{f}(n) = o\left(\frac{1}{|n|}\right)$.

"Little oh" ($\frac{1}{|n|}$) means, given $\epsilon > 0$, for large enough $|n|$, $|o\left(\frac{1}{|n|}\right)| \leq \epsilon/|n|$.

By repeating this argument we see that

Theorem: If $f \in C^k$, then $\widehat{f}(n) = o\left(\frac{1}{|n|^k}\right)$

Now Assume $f \in C^0$ (continuous).

If $|c_n| = |\widehat{f}(n)| \leq \frac{M}{|n|^{1+\delta}}$, $\delta > 0$, then

$\sum c_n e^{inx} = g(x)$ converges uniformly and absolutely to a continuous function $g(x)$. We can integrate term by term to conclude that

$$c_n = \widehat{g}(n).$$

Why does this imply $f(x) = g(x) = \sum c_n e^{inx} ??$

It's a theorem that we have not proved. So it is true that $f(x) = \sum c_n e^{inx}$. This would not be correct if f were not assumed to be continuous. So now we have

$$f(x) = \sum c_n e^{inx}.$$

Now suppose $|\hat{f}(n)| \leq \frac{M}{|n|^{k+l+\delta}}, \delta > 0$.

Then we can repeatedly differentiate term by term
since the differentiated series converges uniformly
and absolutely by the Weierstrass M-test. We have

Theorem Suppose $f \in C^m$ and $|\hat{f}(n)| \leq \frac{M}{|n|^{k+l+\delta}}$
then $f \in C^m, m \leq k$.